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Weakly Hamiltonian-Connected Vertices in Bipartite Tournaments

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We characterize those bipartite tournaments which have a hamiltonian path with given unordered endvertices. Our proof gives rise to a polynomial algorithm to decide the existence of such a path and find one, if it exists.

1. INTRODUCTION

It is well-known that every tournament has a hamiltonian path and every strong tournament has a hamiltonian cycle. For bipartite tournaments the situation is quite different. There exist arbitrarily highly connected bipartite tournaments with no hamiltonian cycle and no hamiltonian path. One such example, taken from [8], is the bipartite tournament $B(k, l, l, k)$ which consists of four disjoint sets of vertices A, B, C, D so that $|A| = |D| = k$ and $|B| = |C| = l$, $l \geqslant k+2$ and all arcs between these sets follow the cycle $A \to B \to C \to D \to A$. It is easy to see that this bipartite tournament is k-connected and that it has no hamiltonian path. Note also that this bipartite tournament is balanced; i.e., it has the same number of vertices in each bipartition class. The reason why there is no hamiltonian cycle in the bipartite tournament above is that it contains no factor—a spanning collection of disjoint cycles. Clearly the existence of a factor is

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a necessary condition for the existence of a hamiltonian cycle. In fact, a strong bipartite tournament has a hamiltonian cycle if and only if it has a factor [4, 5, 7], and a bipartite tournament B has a hamiltonian path if and only if it has a path P such that $B-P$ has a factor [6,7].

In [11] the problem of deciding wether a given tournament, with specified vertices x and y, has a hamiltonian path joining x and y (the order not specified) was solved. It follows from the characterization given in [11] that deciding the existence of such a path for given vertices x and y can be done in polynomial time. In this paper we solve the analogous problem for bipartite tournaments. We give a mathematical characterization and prove that there exists a polynomial algorithm to find such a path if one exists. Perhaps somewhat surprisingly, the characterization is very similar to the tournament case, especially in the case when x and y are adjacent vertices. In the case of nonadjacent vertices we find infinite families of bipartite tournaments without the desired path which have no analogue in the case of tournaments.

2. Terminology and Preliminaries

We shall assume that the reader is familiar with the standard terminolgy on graphs and digraphs and refer the reader to [2, 3].

A *bipartite tournament* B is an orientation of a complete bipartite graph. Let D be a digraph. If there is an arc from a vertex x to a vertex y in D we say that x *dominates* y and use the notation $x \rightarrow y$ to denote this. If A and B are disjoint subsets of $V(D)$ we use the notation $A \not\rightarrow B$ to indicate that there is no arc from A to B (possibly $A=\varnothing$, or $B=\varnothing$). Similarly $A \rightarrow B$ means that $a \rightarrow b$ for every pair $a \in A, b \in B$. In figures we shall sometimes have an arc from one set A of vertices to another set B. This means that all arcs between A and B go from A to B. Note that this is not the same as $A \rightarrow B$!. For $A \subset V(D)-x$ we let $d_A^-(x)$ (respectively $d_A^+(x)$) denote the number of verrices in A dominating x (respectively dominated by x). If x and y are vertices of D and P is a directed path from x to y, we say that P is an (x, y)-path. An (x, y)-hamiltonian path is an (x, y)-path containing all the vertices of D. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing the vertices x and y, $C[x, y]$ denotes the subpath from x to y. By an $\{x, y\}$-path we shall mean a path P which is either an (x, y)- or a (y, x)-path.

A digraph D is *strongly connected* (or just *strong*) if there exists an (x, y)-path and a (y, x)-path in D for any choice of vertices x, y of D. If a digraph is not strong, then we can label its strong components $D_1, \ldots, D_s$, $s \geqslant 2$, so that $D_j \not\rightarrow D_i$ for $1 \leqslant i<j \leqslant s$. In general this labelling is not unique, not

even for bipartite tournaments. On the other hand, the reader can easily verify the following.

PROPOSITION 2.1. *If B is a non-strong bipartite tournament, in which every vertex lies on some cycle, then there is a unique labelling $B_1, ..., B_k$, $k \geqslant 2$, of the strong components of B, satisfying that $B_j \not\to B_i$ for $1 \leqslant i < j \leqslant k$.* ∎

By an *initial* (respectively a *terminal*) component of a digraph D, we mean a strong component with no arcs entering it (respectively leaving it) in D. It is an easy consequence of Proposition 2.1, that if B is non-strong bipartite tournament such that every vertex is on some cycle, then B has precisely one initial and one terminal component. We shall use this fact several times in the sequel.

A *factor* of a digraph D is a spanning subdigraph so that every vertex has in- and out-degree 1, i.e., a collection of disjoint cycles covering the vertices of D.

If P is a path so that $D - V(P)$ has a factor, then we shall say that P is a *path with a cofactor*.

We shall make use of the following results characterizing those bipartite tournaments which have a hamiltonian path and a hamiltonian cycle respectively and giving an algorithm to find a hamiltonian cycle if one exists.

THEOREM 2.2 [6, 7]. *A bipartite tournament B has a hamiltonian path if and only if there exists a path P (possibly trivial or hamiltonian) with a cofactor.*

THEOREM 2.3 [4, 5, 7]. *A strong bipartite tournament has a hamiltonian cycle if and only if it has a factor.*

LEMMA 2.4 [10]. *Given a factor of a bipartite tournament B on n vertices, one can find, in time $O(n^2)$, a collection of cycles $C_1, ..., C_k$, $k \geqslant 1$, such that there is no arc from C_j to C_i for $j > i$.*

Note, that by Lemma 2.4, whenever we are dealing with a collection of cycles $C_1, ..., C_k$, we may assume , posssibly by merging some cycles into one, that there are no arcs from C_j to C_i when $j > i$.

3. THE CHARACTERIZATION

The result obtained in this paper is the following.

THEOREM 3.1. *Let B be a bipartite tournament and x and y distinct vertices of B. B has a $\{x, y\}$-hamiltonian path if and only if B has an*

$\{x, y\}$-path P with a cofactor and B does not satisfy any of the conditions (i–iv) *below*:

(i) *B is not strong and its initial or terminal component* (*or both*) *contains neither x nor y.*

(ii) *B is strong, $B-x$ is not strong and either y belongs to neither the initial nor the terminal component of $B-x$, or y belongs to the initial* (*terminal*) *component of $B-x$ and there is no* (y, x)*-path* $((x, y)$*-path P' with a cofactor.*

(iii) *B is strong, $B-y$ is not strong and either x belongs to neither the initial nor the terminal component of $B-y$, or x belongs to the initial* (*terminal*) *component of $B-y$ and there is no* (x, y)*-path* $((y, x)$*-path*) *P' with a cofactor.*

(iv) *B, $B-x$, and $B-y$ are all strong, x and y are nonadjacent and B belongs to one of the five families of bipartite tournaments $\mathscr{B}_0-\mathscr{B}_4$ in Figs.* 1 *and* 2, *where* $\{x', y'\} = \{x, y\}$.

Proof of Necessity. First observe that the existence of an $\{x, y\}$-path P with a cofactor is necessary, since any $\{x, y\}$-hamiltonian path has that property.

Next observe that if B has such a path P, then, if B is not strong, it has only one initial and one terminal component, and if B is strong, but $B-x$ or $B-y$ is not strong, then again this graph has only one initial and, only terminal component.

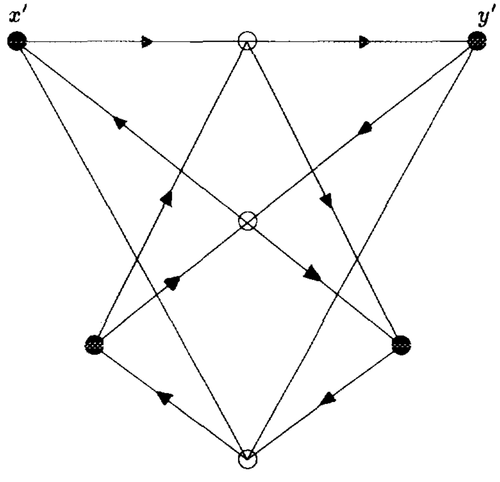

FIG. 1. The family $\mathscr{B}_0$. The edges that are not oriented can be oriented arbitrarily.

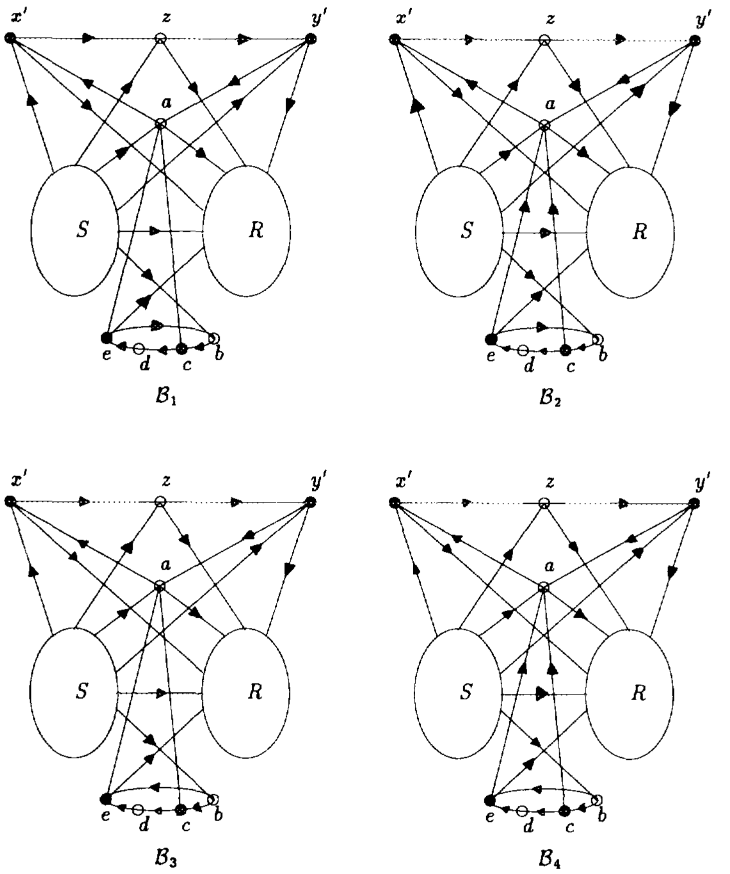

FIG. 2. The infinite families $\mathscr{B}_1$-$\mathscr{B}_4$ of bipartite tournaments. In the figures the arcs drawn indicate the direction of all arcs between the two sets. This does not mean that all vertices in the two sets are adjacent, but only those in opposite bipartition classes. In each of $\mathscr{B}_1$–$\mathscr{B}_4$ we have that the vertices of $R \cup S \cup \{a, b, c, d, e\}$ form a cycle with the order a, R, b, c, d, e, S, a. $R \nrightarrow \{a, x', y', z\} \nrightarrow S$, $b \nrightarrow S$, $R \nrightarrow e$ and $R \nrightarrow S$, $b \rightarrow c \rightarrow d \rightarrow e$, $e \rightarrow z \rightarrow c$, $d \rightarrow y' \rightarrow b$, $x' \rightarrow z \rightarrow y'$, $y' \rightarrow a \rightarrow x'$. The arcs between a and $\{c, e\}$ are either $a \rightarrow \{c, e\}$, or $\{c, e\} \rightarrow a$. In $\mathscr{B}_1$: $e \rightarrow b$, $c \nrightarrow S$, and if $a \rightarrow \{c, e\}$ then $x' \rightarrow \{b, d\}$ and $d \nrightarrow S$. In $\mathscr{B}_2$: $e \rightarrow b$, $R \nrightarrow d$, $\{c, e\} \rightarrow a$, $d \rightarrow x'$, and there exists an arc from c to S. In $\mathscr{B}_3$: $b \rightarrow e$, $\{c, d\} \nrightarrow S$, and if $a \rightarrow \{c, e\}$ then $x' \rightarrow \{b, d\}$. In $\mathscr{B}_4$: $b \rightarrow e$, $R \nrightarrow \{c, d\}$, $\{c, e\} \rightarrow a$, $d \rightarrow x'$, and there exists an arc from c to S.

It is easy to see that if B satisfies any of (i–iii) in Theorem 3.1, or $B \in \mathscr{B}_0$, then there can be no $\{x, y\}$-hamiltonian path. Hence it remains to show that if B is a graph from one of the families $\mathscr{B}_1 - \mathscr{B}_4$ in Fig 2, then B has no $\{x, y\}$-hamiltonian path.

We now consider each of the cases in turn. In each case we shall assume w.l.o.g. that $x' = x$ and $y' = y$:

$B \in \mathscr{B}_1$. We first prove that there is no (y, x)-hamiltonian path in B. Suppose first that $a \to e$. Then, by the definition of $\mathscr{B}_1$, e will separate R from S and $x \to b, x \to d$. Hence there is no arc from $R \cup \{b, d\}$ to x. Hence, on any (y, x)-hamiltonian path, all vertices of R must appear before the vertex e. Now it is easy to see that there can be no such path, because z cannot be picked up.

Now suppose $e \to a$. Then any (y, x)-path must start in R possibly including a. Because d separates R from S any (y, x)-path containing all of R must pick up these vertices before d. Now clearly z cannot be picked up.

Next we rule out the existence of an (x, y)-hamiltonian path. If z is neither the successor of x nor the predecessor of y on such a path, then it will have to make two passes through R, contradicting the fact that d separates R from S: z can only be reached from x or vertices in S and we cannot use the arc $x \to d$ (if it exists) because d separates R from S, so the path must start in R. Hence z is either the successor of x or the predecessor of y on any (x, y)-hamiltonian path. In either case we cannot pick up the vertex a: The path must go to R and take all vertices here before going to d. Furthermore, a has at most one arc to S, namely the arc $a \to e$ is possible, but in that case there is no arc from d to S, by the definition of $\mathscr{B}_1$.

$B \in \mathscr{B}_2$. Here it is easy to see, in a similar way as we did above, that there is no $\{x, y\}$-hamiltonian path. Every path joining these vertices must leave out either a or z.

$B \in \mathscr{B}_3$. We give only the new cases. It is easy to see, by the definition of $\mathscr{B}_3$, that any (y, x)-path containing a and z will have to start with the arcs $y \to a$ and $a \to e$. But if $a \to e$, then $x \to d$ and $x \to b$ by definition of $\mathscr{B}_3$, so the path cannot exist, since after z, the path must go to R and there is no arc from $R \cup \{b, d\}$ to x. The proof that there can be no (x, y)-hamiltonian path is similar to the proof when $B \in \mathscr{B}_1$ (note that e separates R from S).

$B \in \mathscr{B}_4$. This case is left to the reader. The arguments are very similar to the above ones.

This completes the proof of necessity of Theorem 3.1.

Proof of Sufficiency When B Is Not Strong, or $B - x$ or $B - y$ Is Not Strong.

LEMMA 3.2. *Let B be a strong bipartite tournament and u a vertex such that there exists a path P starting (ending) in u such that $B - P$ has a factor. Then B has a hamiltonian path starting (ending) in u.*

Proof. Suppose that P is a path starting in u and ending in some vertex v, so that $B - P$ has a factor $C_1, ..., C_k$, $k \geqslant 0$, where k is minimal.

Suppose $k \geqslant 1$. By Theorem 2.3, we may assume that $C_1, ..., C_k$ form the strong components of $B-P$ and that C_1 is the initial component. Now, if there is an arc from v to C_1, then we get a contradiction to the minimality of k (because P could be extended to include all vertices if C_1 as well), so all arcs beween C_1 and v go to v. Because B is strong, there is some arc from $P-v$ to C_1. Now it is easy to see that all vertices of C_1 can be inserted between two consecutive vertices of P, contradicting the minimality of k. Hence $k=0$ and P is a hamiltonian path. This completes the proof of the lemma. ∎

Using Proposition 2.1, Theorem 2.3, Lemma 3.2, and the fact that B has an $\{x, y\}$-path P with a cofactor, the reader can easily complete the proof that if B is not strong, or $B-x$ or $B-y$ is not strong and B does not satisfy any of (i)–(iii), then B has $\{x, y\}$-hamiltonian path.

The next two sections are devoted to complete the proof of Theorem 3.1 in the case when B, $B-x$, and $B-y$ are all strong. In the following sections, whenever we talk about an (x, y)-path P, we let x^* (respectively, y^*) denote the successor of x (respectively, the predecessor of y) on that path.

4. Hamiltonian Paths with Adjacent Endvertices

In this section we prove that if x and y are adjacent and B, $B-x$, and $B-y$ are all strong, then B has an $\{x, y\}$-hamiltonian path if and only if it has an (x, y)-path with a cofactor. Hence, (i)–(iii) are the only exceptions when x and y are adjacent vertices. We first prove a series of lemmas which finally imply the result in Lemma 4.8.

Definition 4.1. Let D be a digraph with an arc $x \rightarrow y$ and a cycle C disjoint from $\{x, y\}$. An arc $u \rightarrow v$ of C is called a *partner* of the arc $x \rightarrow y$ if $u \rightarrow x$ and $y \rightarrow v$ are both arcs of D.

Lemma 4.2. *Let B a bipartite tournament and let $x \rightarrow y$ be an arc and C a cycle of $B-\{x, y\}$, such that there is an arc from C to* x *and an arc from y to C. Either $x \rightarrow y$ has a partner on C, or there exists an (x, y)-hamiltonian path in $B(V(C) \cup \{x, y\})$.*

Proof. Let $C = u_0 u_1 \cdots u_{2s-1} u_0$, $s \geqslant 2$. The labelling is chosen so that x is adjacent to each u_{2i+1} and y is adjacent to each u_{2i}, $i=0, ..., s-1$. Choose u_{2i+1} and u_{2j} such that $u_{2i+1} \rightarrow x$, $y \rightarrow u_{2j}$ and the length of $C[u_{2i+1}, u_{2j}]$ is as small as possible. If $C[u_{2i+1}, u_{2j}]$ is just an arc, then $x \rightarrow y$ has a partner on C. So suppose that $j \neq i+1 \bmod s$. Now the choice of u_{2i+1} and u_{2j} *implies that* $x \rightarrow u_{2j-1}$ *and* $u_{2j-2} \rightarrow y$. *Thus* $xC[u_{2j-1}, u_{2j-2}]y$ is a hamiltonian path in $B(V(C) \cup \{x, y\})$. ∎

LEMMA 4.3. *Let B be a bipartite tournament. Let P be a path of odd length and C a cycle of B such that $V(P) \cap V(C) = \varnothing$. If each arc $x \to y$ of P has a partner on C, then B has a cycle C' with $V(C') = V(C) \cup V(P)$.*

Proof. Let $P = x_1 x_2 \cdots x_{2t}$, $t \geqslant 1$. We proceed by induction on t. If $t = 1$ the claim is obvious, because $x_1 \to x_2$ has a partner on C, so assume that $t \geqslant 2$. Let $u \to v$ be any partner of $x_1 \to x_2$ on C. Choose i as large as possible such that $x_{2i} \to v$. Clearly $P[x_1, x_{2i}]$ can be inserted in C to give a cycle C^*. Thus if $i = t$ we are done. Otherwise, the claim follows by induction applied to the path $P[x_{2i+1}, x_{2t}]$ and C^*. ∎

LEMMA 4.4. *If B is a bipartite tournament with a cycle C of length $n-2$, such that $V(B) - V(C) = \{x, y\}$, x and y are adjacent and, furthermore, there is an arc from x to C and an arc from C to y, then B has an $\{x, y\}$-hamiltonian path.*

Proof. Let $C = u_0 \cdots u_{2s-1} u_0$ be labelled so that x is adjacent to each u_{2i+1} and y is adjacent to each u_{2i}, $i = 0, ..., s-1$. Choose u_{2i+1} and u_{2j} on C such that $x \to u_{2i+1}$, $u_{2j} \to y$ and $C[u_{2i+1}, u_{2j}]$ is as long as possible. Now either $i = j$, in which case B has an (x, y)-hamiltonian path, or $y \to u_{2i}$ and $u_{2i-1} \to x$, in which case B has the hamiltonian path $yC[u_{2i}, u_{2i-1}]x$. ∎

LEMMA 4.5. *Let B be a bipartite tournament consisting of an (x, y)-path P for some pair of adjacent vertices x, y and a cycle C of length less than $n-2$ disjoint from P. If there exists vertices z, w on P, such that z precedes w on P and there is an arc from z to C and an arc from C to w, then either there is an arc from x^* to C and an arc from C to y^*, or B has an $\{x, y\}$-hamiltonian path.*

Proof. Suppose there is no arc from x^* to C. Then we may assume that $z \neq x$, since otherwise B has an (x, y)-hamiltonian path. If there is an arc from y^* to C, then $B - \{x, y\}$ has a hamiltonian cycle ($P[x^*, y^*]$ can be inserted between two consecutive vertices of C) and the claim follows from Lemma 4.4. So, we may assume that there is no arc from y^* to C. But now clearly all the vertices of C can be inserted somewhere between z and y^*.

Similarly, if there is no arc from C to y^* we can conclude that B has an $\{x, y\}$-hamiltonian path. ∎

LEMMA 4.6. *Let B a bipartite tournament consisting of an (x, y)-path P for some pair of adjacent vertices x, y and a cycle C of length less than $n-2$ disjoint from P. If there exist vertices z, w on P, such that z precedes w on P and there is an arc from z to C and an arc from C to w, then B has an $\{x, y\}$-hamiltonian path.*

Proof. By Lemma 4.5 we may assume that $z = x^*$, $w = y^*$. Now, either all vertices of $P[x^*, y^*] - \{x^*, y^*\}$ have arcs in both directions to C, or all vertices of C may be inserted between some pair of consecutive vertices of P, implying that B has an (x, y)-hamiltonian path. Hence we may assume, by Lemma 4.2, that each arc of $P[x^*, y^*] - \{x^*, y^*\}$ has a partner on C. So, by Lemma 4.3 $B - \{x, y, x^*, y^*\}$ has a hamiltonian cycle, and by Lemma 4.4 there is an $\{x^*, y^*\}$-hamiltonian path in $B - \{x, y\}$. We may assume that this is a (y^*, x^*)-hamiltonian path. Now note that $B - \{x, y\}$ has a hamiltonian cycle: If $x^* \to y^*$ this is clear, and if $y^* \to x^*$ then $B - \{x, y\}$ is strong bipartite tournament which has a factor and, hence, is hamiltonian by Theorem 2.3. Hence our claim follows from Lemma 4.4. ∎

LEMMA 4.7. *Let B be a bipartite tournament consisting of an (x, y)-path P for some pair of adjacent vertices x, y and a cycle C of length less than $n-2$ disjoint from P. If there do not exist vertices z, w on P, such that z precedes w on P and there is an arc from z to C and an arc from C to w, then either there is no arc from C to $P - x$, or there is no arc from $P - y$ to C, or B has an $\{x, y\}$-hamiltonian path.*

Proof. Suppose first that there is an arc from C to y^*. Then the assumption of the lemma implies that there is no arc from x^* to C. Now, either there is no arc from y^* to C, or we can insert $P[x^*, y^*]$ in C and conclude the existence of an $\{x, y\}$-hamiltonian path from Lemma 4.4. But then we easily see that there can be no arc from $P - y$ to C, or all vertices of C could be inserted in P.

Suppose next that there is no arc from C to y^*. Then there can be no arc from C to x^*, or $P[x^*, y^*]$ could be inserted in C and we could finish as above. Now the assumption of the lemma implies that there is no arc from C to $P - x$. ∎

LEMMA 4.8. *Let B be a strong bipartite tournament and x, y be adjacent vertices of B. If $B - x$ and $B - y$ are strong, then B has an $\{x, y\}$-hamiltonian path if and only if B has an $\{x, y\}$-path P with a cofactor.*

Proof. Clearly the condition is necessary. Suppose B has an $\{x, y\}$-path P with a cofactor $C_1, ..., C_k$, $k \geqslant 0$. Among the longest such paths choose P so that k is minimum. W.l.o.g. P is an (x, y)-path. By Theorem 2.3 we may also assume that $C_1, ..., C_k$ form the strong components of B' and that there are no arcs from C_j to C_i for $j > i$. Now, if $k = 1$, then it follows from the fact that $B - x$ and $B - y$ are strong, and from Lemmas 4.7 and 4.6 that B has an $\{x, y\}$-hamiltonian path, contradicting the maximality of P.

Suppose $k \geqslant 2$. Since P is longest possible and since $B - x$ and $B - y$ are strong, it follows from Lemmas 4.6 and 4.7 that C_1 has no arcs to $P - x$

and $P-y$ has no arcs to C_k. Hence all the vertices of B' can be inserted in P: If P is just the arc $x \to y$, then x has an arc to a vertex y'' in C_1 and there is an arc from a vertex x'' on C_k to y, and clearly there is a (y'', x'')-hamiltonian path in B'. So we may assume that P has length at least 3. Let $x \to x^* \to x^{**}$ be the first two arcs of P. Now it is easy to see that the vertices of B' can be inserted between x^* and x^{**}. Hence, $k=0$ and P is a hamiltonian path. ∎

Note that some of the proofs above are only valid for adjacent vertices x and y. The bipartite tournaments in Fig. 1 and 2 show that Lemma 4.8 cannot be extended to hamiltonian paths with the vertices given nonadjacent.

5. Hamiltonian Paths Joining Nonadjacent Vertices

In this section we will complete the proof of Theorem 3.1 by showing that if x and y are nonadjacent and B, $B-x$, $B-y$ are all strong and B does not satisfy (iv) in the theorem, then B has an $\{x, y\}$-hamiltonian path. In what follows B, x and y are fixed and B, $B-x$, and $B-y$ are all strong.

Our first step is to reduce the problem to the case when we have one path of length two between x and y and one cycle C.

Lemma 5.1. *Either B has an $\{x, y\}$-hamiltonian path, or there exists a vertex $z \in B-\{x, y\}$ so that $B-\{x, y, z\}$ is hamiltonian and $x \to z \to y$, or $y \to z \to x$.*

Proof. Let P be an $\{x, y\}$-path with a cofactor $C_1, ..., C_m$, $m \geqslant 0$. Let P be chosen in such a way that m is as small as possible. Assume w.l.o.g. that P is an (x, y)-path. We may assume that $m \geqslant 1$. We shall also assume that m is minimal and that there is no arc from C_j to C_i for $j > i$ (see Lemma 2.4).

Case 1. $m=1$. In this case we may assume that P has at least five vertices. Let $x'(y')$ denote the successor (predecessor) of $x^*(y^*)$ on P.

If there exists an arc from x to C_1 and an arc from C_1 to y, then it is easy to see that B has an (x, y)-hamiltonian path, unless all the vertices of $P[x^*, y]$ have edges to and from C_1 (otherwise all the vertices of C_1 can be inserted between two consecutive vertices of P). By Lemma 4.2 and 4.3, $B-x$ has a hamiltonian cycle C in which y^* is the predecessor of y, and $B-\{x, y, y^*\}$ has a hamiltonian cycle C'. Now, if $y^* \to x$, then B has the (y, x)-hamiltonian path $C[y, y^*]x$. So, we may assume that $x \to y^*$. Then the path $x \to y^* \to y$ and the cycle C' show that the second alternative in the lemma holds.

Now, suppose that there are no arcs from x to C_1 (the case with no arcs C_1 to y is analogous). Since $B-y$ is strong, there is an arc from $P-y$ to C_1. Then it is easy to see that there is an arc from y^* to C_1, or B has an (x, y)-hamiltonian path. From this we conclude that there exists an arc from y to C_1. Also $x^* \to y$, or B has the hamiltonian path $yP[x^*, y^*]\, C_1[v, v']x$, where $v \in C_1$ is a vertex dominated by y^* and v' is the predecessor of v' on C_1.

Suppose first that x^* has arcs to and from C_1. If some vertex after x^* on P has only arcs from C_1, then B has an (x, y)-hamiltonian path. If some vertex u on $P[x', y^*]$ has only arcs to C_1, then all vertices after u, in particular y, must have only arcs to C_1, or C_1 could have been inserted in P. Now, if $y^* \to x$, then B has the hamiltonian path $yC_1[c, c']P[x^*, y^*]x$, where $c' \in C_1$ is a vertex dominating x^* and c' is the predecessor of c on C_1. On the other hand, if $x \to y^*$, then $x \to y^* \to y$ and $B - \{x, y^*, y\}$ is hamiltonian, since the path $P[x^*, y']$ can be inserted in C_1 (x' must have only arcs to C_1, because otherwise the path $P[x', y^*]$ could be inserted in C_1, leaving the path $x \to x^* \to y$). Hence, we may assue that every vertex of $P[x', y^*]$ has arcs to and from C_1, and now it follows from Lemmas 4.2 and 4.3 that either B has an (x, y)-hamiltonian path, or $B - \{x, x^*, y\}$ is hamiltonian.

Hence we may assume, since $B-x$ is strong, that x^* has no arcs to C (if x^* has only arcs to C_1, then it follows from the fact that $B-x$ is strong, that all the vertices of C_1 can be inserted in P). Now $x \to y^*$, or B contains the hamiltonian path $yC_1[c, c']P[x^*, y^*]x$, where $c \in C_1$ is a vertex dominated by y and c' is the predecessor of c on C_1. Starting at y', we now conclude that there are no arcs from $P[x', y']$ to C_1. Otherwise, we either find the desired hamiltonian path, or $P[x^*, y']$ can be inserted in C_1, leaving the path $x \to y^* \to y$. But this means that the path $P[x', y^*]$ can be inserted in C_1, leaving the path $x \to x^* \to y$.

Case 2. $m \geqslant 2$. Recall that we have chosen $C_1, ..., C_m$ so that there is no arc from C_j to C_i for $j > i$. Suppose first that there is an arc from C_m to y. Then, by the minimality of m, there must be an arc from C_m to each vertex on P. Since $B-y$ is strong , there is an arc from $P-y$ to C_1. Now it is easy to see that we can insert all the vertices of $C_1 \cup \cdots \cup C_m$ between two consecutive vertices of P.

So we may assume that there are no arcs from C_m to y and, similarly, that there are no arcs from x to C_1. Since $B-x$ is strong, there is an arc from C_m to $P-x$. Now we conclude that there must be an arc from C_m to x^*, because otherwise the vertices of C_m can be inserted in P. Similarly, there must be an arc from y^* to C_1. But now it is easy to see that all the vertices of C_1 and C_m can be inserted in a path P' from y to x which also contains the vertices of P. This contradicts the minimality of m and the proof is complete. ∎

LEMMA 5.2. *Let z be chosen so that $x \to z \to y$ and $B - \{x, y, z\}$ has hamitonian cycle $C = u_0 u_1 \cdots u_{2k-1} u_0$. Let C be labelled so that z is adjacent to u_{2i} for $i = 0, ..., k-1$. If z has at least one arc to and from C, then B satisfies at least one of the following conditions*:

I. *B has an $\{x, y\}$-hamiltonian path,*

II. *The in-neighbours of z on C are $u_{2i}, u_{2(i+1)}, ..., u_{2(i+s)}$ for some i and some $0 \leqslant s \leqslant k-1$,*

III. *C has $4p$ vertices for some p and the neighbours of z on C alternate around C between in- and out-neighbours.*

Proof. Clearly, I will hold unless every predecessor of an out-neighbour of z on C is dominated by y and every successor of an in-neighbour of z on C dominates x. So, assume that this is the case below and suppose that II does not hold. Then there exist vertices u_{2i+1} and u_{2j+1}, $i \neq j$, on C such that $u_{2i} \to z \to u_{2i+2}$ and $u_{2j} \to z \to u_{2j+2}$. Now, either $u_{2j+2} \to u_{2j+1}$ and $u_{2j+1} \to u_{2i}$, or B has one of the hamiltonian paths.

$$yu_{2i+1} C[v_{2j+2}, u_{2i}]\, zC[u_{2i+2}, u_{2j+1}]\, x,$$

$$yC[u_{2i+1}, u_{2j}]\, zC[u_{2j+2}, u_{2i}]\, u_{2j+1} x.$$

Similarly, $u_{2i+2} \to u_{2j+1}$ and $u_{2i+1} \to u_{2j}$.

Next, we conclude that $x \to u_{2i-1}$, or $yC[u_{2i+1}, u_{2j+1}]\, u_{2i} z\, C[u_{2j+2}, u_{2i-1}]\, x$ is a hamiltonian path.

From this we get that $z \to u_{2i-2}$, or $xC[u_{2i-1}, u_{2i-2}]\, zy$ is a hamiltonian path. Similarly, $u_{2j+3} \to y$ and $u_{2j+4} \to z$. Thus, we have a new vertex u_{2r+1} on $C[u_{2j+1}, u_{2i+1}]$ so that $u_{2r} \to z \to u_{2r+2}$. Containing this way, it is easy to see that either I or III holds. ∎

LEMMA 5.3. *Let $x \to z \to y$ be a path so that $B - \{x, y, z\}$ has hamiltonian cycle $C = u_0 u_1 \cdots u_{2k-1} u_0$, $k \geqslant 4$. If B satisfies* III *in Lemma* 5.2, *then B has an $\{x, y\}$-hamiltonian path.*

Proof. Let u_{2i+1} and u_{2j+1}, $i \neq j$, be any pair of vertices of C so that $u_{2i} \to z \to u_{2i+2}$ and $u_{2j} \to z \to u_{2j+2}$. As in the proof of Lemma 5.2, we may assume that B has the following arcs: $u_{2i+2} \to u_{2j+1}$, $u_{2i+1} \to u_{2j}$, $u_{2j+2} \to u_{2i+1}$, $u_{2j+1} \to u_{2i}$, $x \to u_{2i-1}$, $z \to u_{2i-2}$, $u_{2j+3} \to y$, $u_{2j+4} \to z$.

First observe that $u_{2j+3} \to u_{2i}$, since otherwise we have the hamiltonian path $xzu_{2j+2} C[u_{2i+1}, u_{2j+1}]\, u_{2i} C[u_{2j+3}, u_{2i-1}]\, y$. ($u_{2i-1} \to y$, by the argument above, because $u_{2i-4} \to z \to u_{2i-2}$.)

If $u_{2i+2} \to u_{2j+3}$, then we have the hamiltonian path $xC[u_{2i+3}, u_{2j+2}]\, u_{2i+1} u_{2i+2} C[u_{2j+3}, u_{2i}]\, zy$ ($x \to u_{2i+3}$, by the argument above, because $u_{2i+4} \to z \to u_{2i+6}$. So $u_{2j+3} \to u_{2i+2}$ and, similarly, $u_{2i+3} \to u_{2j}$ and $u_{2i+3} \to u_{2j+2}$.

Now, if $k > 4$, then B has the hamiltonian path

$$xzu_{2j+2}u_{2j+3}C[u_{2i+2}, u_{2j+1}]\, u_{2i}u_{2i+1}C[u_{2j+4}, u_{2i-1}]\, y.$$

So suppose $k = 4$. Then, $u_{2i+3} \to u_{2i}$, or we have the hamiltonian path $xu_{2j+3}u_{2i+2}u_{2j+1}u_{2i}u_{2i+3}u_{2j+2}u_{2i+1}u_{2j}zy$. Similarly, $u_{2j+3} \to u_{2j}$. Now, we have the hamiltonian path $xu_{2i+3}u_{2i}u_{2i+1}u_{2i+2}u_{2j+1}u_{2j+2}u_{2j+3}u_{2j}zy$. ▮

LEMMA 5.4. *Let z be chosen so that $\{x, y, z\}$ induce an $\{x, y\}$-path and B—$\{x, y, z\}$ has a hamiltonian cycle C. Either B has an $\{x, y\}$-hamiltonian path, or we may assume, by choosing a new vertex z if necessary, that either $d_C^+(y) = d_C^+(z)$ or $d_C^-(x) = d_C^-(z)$.*

Proof. We may assume w.l.o.g. that $x \to z \to y$. Suppose that B has no $\{x, y\}$-hamiltonian path. First observe that

$$d_C^-(x) \geqslant d_C^-(z) \qquad \text{and} \qquad d_C^+(y) \geqslant d_C^+(z). \tag{1}$$

The first holds because x is dominated by the successor of any in-neighbour of z, or B has the desired path. The second claim is proved similarly. Hence we may assume that z has arcs to and from C. Let z' be a vertex of C so that z is dominated by (dominates) the predecessor (successor) of z'. Clearly, $y \to z' \to x$ and $C' = (C - z') \cup \{z\}$ is a hamiltonian cycle in $B - \{x, y, z'\}$.

Let $2k$ be the number of vertices on C. If $d_{C'}^+(z') + d_{C'}^-(x) > k$, or $d_{C'}^+(y) + d_{C'}^-(z') > k$ then it is easy to see that B has a (y, x)-hamiltonian path. Hence, we may assume

$$d_{C'}^+(z') + d_{C'}^-(x) \leqslant k \tag{2}$$

and

$$d_{C'}^+(y) + d_{C'}^-(z') \leqslant k. \tag{3}$$

From, (1),(2), and (3) we easily get that

$$k = d_C^+(z) + d_C^-(z) \leqslant d_C^+(y) + d_C^-(x) \leqslant k + 2. \tag{4}$$

If the second inequality in (4) holds with equality, then we get equality in each of (2) and (3) and, hence, we may use z' instead of z. Otherwise, by (1), either $d_C^+(y) = d_C^+(z)$ or $d_C^-(x) = d_C^-(z)$. ▮

LEMMA 5.5. *B has an $\{x, y\}$-hamiltonian path unless B belongs to one of the families $\mathscr{B}_0 - \mathscr{B}_4$ in Figs. 1 and 2.*

Proof. By Lemma 5.1 and Lemma 5.4, we may assume w.l.o.g. that there exists a vertex z so that $x \to z \to y$ and $C = u_0 u_1 \cdots u_{2k-1} u_0$ is a hamiltonian cycle of $B - \{x, y, z\}$, and $d_C^+(y) = d_C^+(z)$.

If $k = 2$, then it is easy to see that the desired path exists, unless B is one of the graphs in $\mathscr{B}_0$. So, we may assume that $k \geqslant 3$.

Note that z has at least one arc to and from C because $B - x$ and $B - y$ are strong. Hence, by Lemma 5.2 and Lemma 5.3, we may assume that

$$\{u_1, ..., u_{2r-2}\} \not\mapsto z \not\mapsto \{u_{2r+1}, ..., u_{2k-1}\} \qquad \text{for some } r, \quad 1 \leqslant r < k. \quad (5)$$

Since $d_C^+(y) = d_C^+(z)$, we have $d_C^+(y) = r$, and y dominates precisely $\{u_0, u_2, ..., u_{2r-2}\}$, or B has the desired path. So we have

$$\{u_0, ..., u_{2r-2}\} \not\mapsto y \not\mapsto \{u_{2r}, ..., u_{2k-2}\}. \qquad (6)$$

Similarly,

$$x \not\mapsto \{u_{2r+2}, ..., u_0\}. \qquad (7)$$

If $u_{2j+1} \to u_0$ for some $j \in \{1, 2, ..., r-2\}$, then we have the hamiltonian path $yu_{2j+2}u_{2j+3} \cdots u_{2k-1}zu_1 \cdots u_{2j+1}u_0x$. Hence, we may assume that

$$\{u_1, u_3, ..., u_{2r-3}\} \not\mapsto u_0. \qquad (8)$$

Similarly, we conclude that

$$u_0 \not\mapsto \{u_{2r+3}, ..., u_{2k-1}\}. \qquad (9)$$

$$\text{If} \quad u_0 \to u_{2r+1} \quad \text{then} \quad x \to u_{2j} \quad \text{for} \quad 1 \leqslant j \leqslant r. \qquad (10)$$

Otherwise, $yu_0u_{2j+1} \cdots u_{2k-1}zu_1 \cdots u_{2j}x$ is a hamiltonian path.

$$u_{2j} \to u_{2i+1} \qquad \text{for} \quad 0 \leqslant i \leqslant r-2, \quad r+1 \leqslant j \leqslant k-1. \qquad (11)$$

Otherwise, there exists an arc $u_{2i+1} \to u_{2j}$ for some $i \in \{0, 1, ..., r-2\}$ and $j \in \{r+1, ..., k-1\}$, and we have the hamiltonian path $yu_{2i+2} \cdots u_{2j-1}zu_1 \cdots u_{2i+1}u_{2j} \cdots u_0x$.

$$u_{2j+1} \to u_{2i} \qquad \text{for} \quad 1 \leqslant i \leqslant r-2, \quad r \leqslant j \leqslant k-1. \qquad (12)$$

Otherwise, there exists an arc $u_{2i} \to u_{2j+1}$, for some $i \in \{1, ..., r-2\}$ and $j \in \{r, ..., k-1\}$, and then, using (8), we have the hamiltonian path $xzu_1 \cdots u_{2i}u_{2j+1} \cdots u_0u_{2i+1} \cdots u_{2j}y$.

$$u_{2r-2} \not\mapsto \{u_{2r+3}, ..., u_{2k-1}\}. \qquad (13)$$

Otherwise, there exists an arc $u_{2r-2} \to u_{2j+1}$, for some $j \in \{r+1, ..., k-1\}$, and we have the hamiltonian path $yu_0 \cdots u_{2r-2}u_{2j+1} \cdots u_{2k-1}zu_{2r-1} \cdots u_{2j}x$.

$$\text{If} \quad u_{2r-1} \to u_0 \quad \text{then} \quad u_{2r+1} \to u_0; \qquad \text{if} \quad u_0 \to u_{2r+1} \quad \text{then} \quad u_0 \to u_{2r-1}. \tag{14}$$

Otherwise, the in-neighbours of u_0 on the cycle $u_1 \cdots u_{2k-1}zu_1$ are not consecutive, and, hence, we have an $\{x, y\}$-hamiltonian path by Lemma 5.2 and Lemma 5.3 applied to the path yu_0x and the cycle $u_1 \cdots u_{2k-1}zu_1$.

$$\text{If} \quad x \to u_{2r} \quad \text{then} \quad u_{2r-1} \not\to \{u_{2r+4}, ..., u_{2k-2}\}. \tag{15}$$

Otherwise, if $x \to u_{2r}$ and $u_{2r-1} \to u_{2i}$ for some $i \in \{r+2, ..., k-1\}$ then we have the hamiltonian path $xu_{2r} \cdots u_{2i-1}u_0 \cdots u_{2r-1}u_{2i} \cdots u_{2k-1}zy$.

$$\text{If} \quad u_0 \to u_{2r+1}, \quad \text{then} \quad u_{2r} \not\to \{u_{2r+3}, ..., u_{2k-1}\}. \tag{16}$$

Suppose not and let $j \geq r+1$ be chosen so that $u_{2r} \to u_{2j+1}$. Now if $u_{2j} \to u_{2k-1}$, then B has the hamiltonian path $yu_0u_{2r+1} \cdots u_{2j}u_{2k-1}zu_1 \cdots u_{2r}u_{2j+1} \cdots u_{2k-2}x$. So $u_{2k-1} \to u_{2j}$, and B has the hamiltonian path $yu_0u_{2r+1} \cdots u_{2j-1}zu_1 \cdots u_{2r}u_{2j+1} \cdots u_{2k-1}u_{2j}x$.

Let $R = \{u_1, u_2, ..., u_{2r-3}\}$ and $S = \{u_{2r+2}, u_{2r+3}, ..., u_{2k-1}\}$. Whenever we say that $B \in \mathscr{B}_i$ below, it is understood that we use the identifications of vertices, $x' = x$, $y' = y$, $a = u_0$, $b = u_{2r-2}$, $c = u_{2r-1}$, $d = u_{2r}$, $e = u_{2r+1}$.

$$\text{If} \quad u_{2r+1} \to u_{2r-2} \quad \text{then} \quad u_{2r-1} \to u_{2i} \qquad \text{for some} \quad i \in \{r+1, ..., k-1\}. \tag{17}$$

Suppose there is no arc from u_{2r-1} to S. Then $u_0 \to u_{2r+1}$, or $B \in \mathscr{B}_1$. Thus, by (10) and (14), $u_0 \to u_{2r-1}$, $x \to u_{2r}$, and $x \to u_{2r-2}$, and again $B \in \mathscr{B}_1$.

$$\text{If} \quad u_{2r+1} \to u_{2r-2} \quad \text{then} \quad u_{2j+1} \to u_{2r} \qquad \text{for some} \quad j \in \{0, ..., r-2\}. \tag{18}$$

Suppose there is no arc from R to u_{2r}. If $x \to u_{2r}$, then $u_{2r-1} \to u_{2r+2}$, by (15) and (17). Now we have one of the hamiltonian paths $xu_{2r}u_1 \cdots u_{2r-1}u_{2r+2} \cdots u_{2k-1}u_0u_{2r+1}zy$, $xu_{2r}u_{2r+1}u_0 \cdots u_{2r-1}u_{2r+2} \cdots u_{2k-1}zy$. So, $u_{2r} \to x$, and then $u_{2r+1} \to u_0$, by (10); but this implies that $B \in \mathscr{B}_2$.

$$\text{If} \quad u_{2r-2} \to u_{2r+1} \quad \text{then} \quad u_{2i} \to u_{2r-1} \text{ implies } u_{2i+1} \to u_{2r} \quad \text{for} \quad i \in \{r+1, ..., k-1\}. \tag{19}$$

Otherwise, the arc $u_{2r-1} \to u_{2r}$ could be inserted in $C[u_{2r+2}, u_{2k-1}]$ and this would give a new cycle in which the in-neigbours of z are not consecutive and, hence, B has an $\{x, y\}$-hamiltonian path, by Lemmas 5.2 and 5.3.

$$\text{If} \quad u_{2r-2} \rightarrow u_{2r+1} \quad \text{then} \quad u_{2r} \rightarrow u_{2j+1} \text{ implies } u_{2r-1} \rightarrow u_{2j} \quad \text{for} \quad j \in \{0, ..., r-2\}. \tag{20}$$

Otherwise, the arc $u_{2r-1} \rightarrow u_{2r}$ could be inserted in $C[u_0, u_{2r-3}]$ and this would give a new cycle so that u_{2r} is the predecessor of u_{2j+1} and $u_{2r} \rightarrow y$ and $z \rightarrow u_{2j+1}$ are both arcs. Hence, B has an (x, y)-hamiltonian path.

$$\text{If} \quad u_{2r-2} \rightarrow u_{2r+1} \quad \text{then} \quad u_{2r-1} \rightarrow u_{2i} \quad \text{for some} \quad i \in \{r+1, ..., k-1\}. \tag{21}$$

Suppose that there is no arc from u_{2r-1} to S. Then, by (19), there is no arc from u_{2r} to S and, hence, we have $B \in \mathscr{B}_3$.

$$\text{If} \quad u_{2r-2} \rightarrow u_{2r+1} \quad \text{then} \quad u_{2j+1} \rightarrow u_{2r} \quad \text{for some} \quad j \in \{0, ..., r-2\}. \tag{22}$$

Suppose there is no arc from R to u_{2r}. Then, by (20), there is no arc from R to u_{2r-1}. Now we can prove that $B \in \mathscr{B}_4$ in a similar way as we proved $B \in \mathscr{B}_2$ in the proof of (18).

$$\exists i \in \{r+1, ..., k-1\}, j \in \{0, ..., r-2\} \quad \text{such that} \quad u_{2r-1} \rightarrow u_{2i}, u_{2j+1} \rightarrow u_{2r}. \tag{23}$$

This is a direct consequence of (17), (18), (21) and (22).

$$u_{2r-1} \nrightarrow \{u_{2r+4}, ..., u_{2k-2}\}, \qquad u_{2r-1} \rightarrow u_{2r+2}. \tag{24}$$

To see this, let $u_{2j+1}, j \in \{0, ..., r-2\}$, be chosen so that $u_{2j+1} \rightarrow u_{2r}$. Now, if $u_{2r-1} \rightarrow u_{2i}$ for some $i \in \{r+2, ..., k-1\}$, then we have the hamiltonian path $yu_{2j+2} \cdots u_{2r-1} u_{2i} \cdots u_{2k-1} zu_1 \cdots u_{2j+1} u_{2r} \cdots u_{2i-1} u_0 x$.

$$u_0 \rightarrow \{u_{2r-1}, u_{2r+1}\}. \tag{25}$$

That $u_0 \rightarrow u_{2r+1}$ is seen above, because, if $u_{2r+1} \rightarrow u_0$, then we would get a hamiltonian path with the same structure as in (24). Now it follows from (14) that $u_0 \rightarrow u_{2r-1}$.

$$\{u_2, ..., u_{2r}\} \nrightarrow x. \tag{26}$$

This follows from (10) and (25).

$$u_{2r-3} \rightarrow u_{2r}, \qquad \{u_1, ..., u_{2r-5}\} \nrightarrow u_{2r}. \tag{27}$$

Otherwise, if $u_{2j+1} \to u_{2r}$ for some $j \in \{0, ..., r-3\}$, then we have the hamiltonian path $xu_{2j+2} \cdots u_{2r-1} u_{2r+2} \cdots u_0 u_1 \cdots u_{2j+1} u_{2r+1} zy$.

Now we have the hamiltonian path $xu_{2r-2} u_{2r-1} u_{2r+2} \cdots u_0 u_1 \cdots u_{2r-3} u_{2r} u_{2r+1} zy$ and the proof is complete. ∎

The proof of Theorem 3.1 is complete. ∎

6. A Polynomial Algorithm for Testing the Existence of a Hamiltonian Path Joining Prescribed Vertices

Theorem 6.1. *There is an $O(n^{5/2}/(\log n)^{1/2})$ algorithm for deciding if distinct vertices x and y in B are connected by a hamiltonian path, and for finding such a path if one exists.*

Proof. We will not give such an algorithm in detail, but rather we will argue that such an algorithm is inherent in our proof. First we need to check that the condition that B contains an $\{x, y\}$-path P with a cofactor is satisfied.

Suppose first that x and y are adjacent. Add the opposite arc between x and y so that they form a 2-cycle. It is clear that now the desired hamiltonian path exists only if this new digraph has a factor containing one of the arcs $x \to y$, $y \to x$. We show how to decide the existence of a factor containing the arc $x \to y$; the other case is handled analogously.

Delete all arcs into y and all arcs out of x, other than the arc $x \to y$, and then try to find one perfect matching from X to Y and another from Y to X. If both of these matchings exist, then their union is a factor containing the arc $x \to y$. Otherwise, if one of these matchings does not exist, there is no factor containing the arc $x \to y$. If there is no factor containing either $x \to y$ or $y \to x$, then x and y are not weakly hamiltonian-connected.

Similarly, we can check the existence of the path P when x and y are nonadjacent, by adding a new vertex z and both of the arcs $x \to z$ and $z \to y$ and then testing for a factor in this bipartite digraph and that obtained by reversing the two new arcs. These tests can be completed in time $O(n^{5/2}/(\log n)^{1/2})$, which is the best known complexity of finding a maximum matching in a bipartite graph [1].

Assume in the following arguments that we have found an $\{x, y\}$-path P with a cofactor. It can be checked in time $O(n^{5/2}/(\log n)^{1/2})$ whether B satisfies any of (i)–(iii). Again it is the complexity of testing for the existence of a path and a factor that dominates the other steps. If B does not satisfy any of (i)–(iii) and either B is not strong, or $B-x$ or $B-y$ is not strong, then we know that the desired path exists and it can be found as follows: First note that the proof of Lemma 3.2 can easily be turned into an $O(n^2)$ algorithm to find a hamiltonian path starting (ending) in u, provided that we are given any path starting (ending) in u which satisfies the condition of the lemma.

We will show how to find the desired path in the case when $B-x$ is not strong and we have an (x, y)-path P such that $B-P$ has a factor and y is in the terminal component of $B-P$. Let $B_1, ..., B_k$ denote the strong components of $B-x$. For each B_i we let P_i denote the intersection of P with B_i. Each P_i is a path and B_i-P_i has a factor. Hence, by the above, we can find a hamiltonian path in B_i starting in the same vertex as P_i. Now, using Proposition 2.1, it is easy to see that these hamiltonian paths can be put together into one (x, y)-path containing all the vertices of those B_i that intersect P. Since x has an arc to B_1 and, by Proposition 2.1, B_j has no arcs to B'_j for $j'<j$, it is easy to see that this path can be extended to the desired hamiltonian path (each component that does not meet P has a hamiltonian cycle, by Theorem 2.3, and this can be found in time $O(n^2)$, by the algorithm of Lemma 2.4).

The remaining cases are proved similarly and the proofs are left to the reader. It remains to prove that we can find the desired path in the case when B is strong and $B-x$, $B-y$ are strong and B does not satisfy (iv) in Theorem 3.1. This follows from a close inspection of the proof in Sections 4 and 5. Most of the steps taken here find constructively either the desired path or the structure of one of the families $\mathscr{B}_0$-$\mathscr{B}_4$. These and the rest of the steps can easily be converted into an $O(n^2)$ algorithm. Thus the complexity of the whole algorithm is dominated by the complexity of finding a maximum matching in a bipartite graph. ∎

7. CONCLUDING REMARKS

First we point out that all the results in this paper are valid for semicomplete bipartite digraphs as well (a *semicomplete bipartite digraph* is a bipartite digraph with bipartition (X, Y) so that each pair of vertices $x \in X$, $y \in Y$ is adjacent). This can be seen by checking the proofs carefully (we never use the assumption that there is only one arc between two vertices).

In [11] it is shown that every 4-connected semicomplete digraph (all pairs of vertices are adjacent and there may be directed cycles of length 2) is strongly hamiltonian connected; i.e., for any pair of vertices x and y there is an (x, y)-hamiltonian path. This is best possible. It is natural to ask wether restults of the same type as those we obtained in this paper for weak hamiltonian connectivity could be obtained for strong hamiltonian connectivity in the case of semicomplete bipartite digraphs. We believe that this is possible and we conjecture that every 4-connected semicomplete bipartite digraph B has an (x, y)-hamiltonian path for any choice of distinct vertices x and y, provided B has an (x, y)-path P with a cofactor.

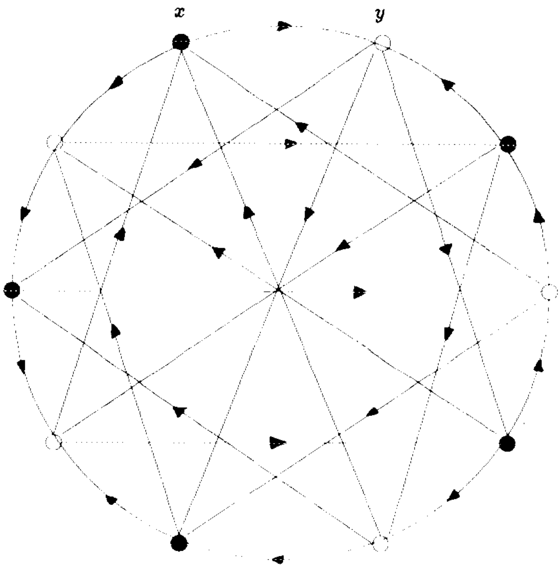

FIG. 3. A 2-connected bipartite tournament B containing an (x, y)-path P with a cofactor (here P is just the arc $x \to y$). B has no (x, y)-hamiltonian path.

We do not have an example yet to show that this would be best possible in terms of connectivity, but the example in Fig. 3 shows that 2-connected is not sufficient. This example can be generalized to an infinite family by repeatedly adding a new pair of vertices x', y' so that $x' \to y'$, $x' \to x$, $y \to y'$ and all other new arcs go to x' and from y'. It is not hard to see that the digraph obtained in this way is 2-connected and has no (x', y')-hamiltonian path.

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